HAHN FIELD REPRESENTATION OF A. ROBINSON'S ASYMPTOTIC NUMBERS

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Abstract

Let ${}^*\mathbb{R}$ be a nonstandard extension of $\mathbb{R}$ and ρ be a positive infinitesimal in ${}^*\mathbb{R}$. We show how to create a variety of isomorphisms between A. Robinson's field of asymptotic numbers ${}^\rho\mathbb{R}$ and the Hahn field $\widehat{{}^\rho\mathbb{R}}(t^{\mathbb{R}})$, where $\widehat{{}^\rho\mathbb{R}}$ is the residue class field of ${}^\rho\mathbb{R}$. Then, assuming that ${}^*\mathbb{R}$ is fully saturated we show that $\widehat{{}^\rho\mathbb{R}}$ is isomorphic to ${}^*\mathbb{R}$ and so ${}^\rho\mathbb{R}$ contains a copy of ${}^*\mathbb{R}$. As a consequence (that is important for applications in non-linear theory of generalized functions) we show that every two fields of asymptotic numbers corresponding to different scales are isomorphic.

Key words: Robinson's non-standard numbers, Robinson's asymptotic numbers, Robinson's valuation field, non-archimedean field, Hahn field, Levi-Civita series, Laurent series, Colombeau's new generalized functions, asymptotic functions.
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1 Introduction

The main purpose of this article is to study in detail some properties of Abraham Robinson's field of *asymptotic numbers* ${}^{\rho}\mathbb{R}$ (known also as *Robinson's valuation field*) and its residue class field $\widehat{{}^{\rho}\mathbb{R}}$ (A. Robinson ([23] and A.H. Lightstone and A. Robinson [13]). In this sense the article is a continuation of the handful of papers written on the subject (known as *non-standard asymptotic analysis*), including: W.A.J. Luxemburg [14], Li Bang-He [11], B. Diarra [3] and V. Pestov [20]. Our emphasis, however, is on those properties of ${}^{\rho}\mathbb{R}$ and $\widehat{{}^{\rho}\mathbb{R}}$ which seem to be of importance for some more recent applications of ${}^{\rho}\mathbb{R}$ to the non-linear theory of generalized functions and partial differential equations: M. Oberguggenberger [17]-[18], M. Oberguggenberger and T. Todorov [19], T. Todorov [24]-[26]. In particular, the complex version ${}^{\rho}\mathbb{C}$ of the field ${}^{\rho}\mathbb{R}$ could be viewed as a "non-standard counterpart" of Colombeau's ring of generalized numbers $\overline{\mathbb{C}}$ (J.F. Colombeau [2]). Similarly, the algebra of asymptotic functions ${}^{\rho}\mathcal{E}(\Omega)$ (M. Oberguggenberger and T. Todorov [19]) can be viewed as a "non-standard counterpart" of a typical algebra of generalized functions $\mathcal{G}(\Omega)$ in Colombeau's theory, where the constant functions in ${}^{\rho}\mathcal{E}(\Omega)$ are exactly the asymptotic numbers in ${}^{\rho}\mathbb{C}$ (T. Todorov [26]). We should mention that ${}^{\rho}\mathbb{C}$ is an algebraically closed Cantor complete non-Archimedean field while $\overline{\mathbb{C}}$ is a non-Archimedean ring with zero divisors. Thus, the involvement of the non-standard analysis results to an essential improvement of the properties of the generalized scalars, compared with the original Colombeau's theory of generalized functions.

2 Preliminaries: Non-Archimedean Fields

In this section we cover various preliminary material. Given a (totally) ordered ring, we define *absolute value* in the usual way: $|x| = \max\{x, -x\}$. *Intervals* and the *order topology* are also defined as usual. Unless specified otherwise, we let every ordered structure be topologized with the order topology. We assume the reader is familiar with the basics of *non-archimedean fields*. If $\mathbb{K}$ is a nonarchimedean field and $x, y \in \mathbb{K}$, we write $x \approx y$ to mean that $x - y$ is infinitesimal. For more details, see T. Lightstone and A. Robinson [13].

In any ordered field, we define the usual notions of *fundamental* or *Cauchy* sequences and *convergent* sequences, with respect to absolute value. In a non-

archimedean field, a sequence (a_n) is fundamental if and only if the sequence of differences $(a_{n+1} - a_n)$ converges to 0. This fact simplifies the theory of series in a sequentially (Cauchy) complete non-archimedean field: a series converges iff the terms approach 0, and there is no such thing as conditional convergence.

Definition 2.1 (Cantor Completeness)*: Let κ be an infinite cardinal. An ordered field is called* ***κ-Cantor complete*** *if every collection of fewer than κ closed intervals with the finite intersection property (F.I.P.) has nonempty intersection. An ordered field is simply called* **Cantor complete** *if it is $\aleph_1$-Cantor complete (where $\aleph_1$ denotes the successor of $\aleph_0 = \text{card}(\mathbb{N})$). This means every nested sequence of closed intervals has nonempty intersection.*

It is easy to show that a Cantor complete ordered field must be sequentially complete. Two counterexamples to the converse are described in the discussion of Hahn fields later in this section.

Recall that the usual definition of κ-saturation in non-standard analysis is obtained by replacing "closed interval" by "internal set" in the above definition. There is also a model-theoretic or first-order definition of κ-saturation, in which "internal set" is replaced by "definable set" (and which for *real-closed* fields is equivalent to replacing "closed interval" by "open interval"). This is weaker than the non-standard notion, but still much stronger than κ-Cantor completeness.

Example 2.1 (Non-Standard Real Numbers) *A typical examples for non-archimedean fields are the non-standard extensions of $\mathbb{R}$ in the framework of the non-standard analysis : A set ${}^*\mathbb{R}$ is called a* ***non-standard extension*** *of $\mathbb{R}$ if $\mathbb{R} \subsetneqq {}^*\mathbb{R}$ and there exists a mapping $*$ from $V(\mathbb{R})$ into $V({}^*\mathbb{R})$ which maps $\mathbb{R}$ at ${}^*\mathbb{R}$, and which satisfies the Transfer Principle (A. Robinson [22]). Here $V(\mathbb{R})$ and $V({}^*\mathbb{R})$ denote the superstructures on $\mathbb{R}$ and ${}^*\mathbb{R}$, respectively. Thus, by definition, any non-standard extension ${}^*\mathbb{R}$ of $\mathbb{R}$ is a totally ordered real closed field extension of $\mathbb{R}$ (with respect to the operations inherited from $\mathbb{R}$), since $\mathbb{R}$ is a totally ordered real closed field. Also, ${}^*\mathbb{R}$ must be a* ***non-archimedean field*** *as a proper field extension of $\mathbb{R}$. For more details on non-standard analysis we refer to A. Robinson [22] and T. Lindstrøm [12], where the reader will find many other references to the subject. For a really brief introduction to the subject - see the Appendix in T. Todorov [25].*

Definition 2.2 (Valuation) *Let $\mathbb{K}$ be an ordered field. A function v from $\mathbb{K}$ to $\mathbb{R} \cup \{\infty\}$ is called a **valuation** on $\mathbb{K}$ provided (for any $x, y \in \mathbb{K}$):*

(a) $v(x) = \infty$ if and only if $x = 0$

*(b) $v(xy) = v(x) + v(y)$ (**logarithmic property**)*

*(c) $v(x + y) \geq \min\{v(x), v(y)\}$ (**non-archimedean property**)*

*(d) $|x| < |y|$ implies $v(x) \geq v(y)$ (**convexity** or **compatibility with the ordering on** $\mathbb{K}$). If v is a valuation on $\mathbb{K}$, the pair $(\mathbb{K}, v)$ is called a **valuation field**. (Here it is understood that $x \leq \infty$ and $x + \infty = \infty$, for every x in $\mathbb{R} \cup \{\infty\}$.)*

We give $\mathbb{R}\cup\{\infty\}$ the usual order topology: all ordinary open intervals and all intervals of the form $(a, \infty]$ are basic open sets. If v is a valuation then it is easy to show that $v(1) = 0, v(-x) = v(x), v(1/x) = -v(x)$ whenever $x \neq 0$, and $v(x) \neq v(y)$ implies $v(x + y) = \min\{v(x), v(y)\}$. For additional details on valuations, we refer the reader to P. Ribenboim [21] or A.H. Lightstone and A. Robinson [13].

The **trivial valuation** on $\mathbb{K}$ is the one defined by $v(x) = 0$ for every non-zero $x \in \mathbb{K}$. This is the only possible valuation on an archimedean field.

Given a valuation field $(\mathbb{K}, v)$, we can define several important structures: $\mathcal{R}_v(\mathbb{K}) = \{x \in \mathbb{K} \mid v(x) \geq 0\}$ is a convex subring of $\mathbb{K}$ called the **valuation ring** of $(\mathbb{K}, v)$. $\mathcal{I}_v(\mathbb{K}) = \{x \in \mathbb{K} \mid v(x) > 0\}$ is also convex and is the unique maximal ideal in $\mathcal{R}_v(\mathbb{K})$, called the **valuation ideal** of $(\mathbb{K}, v)$. And $\widehat{\mathbb{K}} = \mathcal{R}_v(\mathbb{K})/\mathcal{I}_v(\mathbb{K})$ is an ordered field called the **residue class field** of $(\mathbb{K}, v)$. Also, we let $\mathcal{U}_v(\mathbb{K}) = \mathcal{R}_v(\mathbb{K})\backslash\mathcal{I}_v(\mathbb{K}) = \{x \in \mathbb{K} \mid v(x) = 0\}$, the multiplicative group of units of $\mathcal{R}_v(\mathbb{K})$. The elements of the multiplicative group $\mathcal{I}_v(\mathbb{K}_+) = \{x \in \mathbb{K}_+ \mid v(x) > 0\}$ are the **scales** of $\mathbb{K}$. Finally, the **valuation group** of $(\mathbb{K}, v)$, denoted G_v, is the image of $\mathbb{K} \setminus \{0\}$ under v, a subgroup of $\mathbb{R}$.

A valuation v on a field $\mathbb{K}$ induces a metric d, called a **valuation metric**, by the rule $d(a, b) = e^{-v(a-b)}$, with the understanding that $e^{-\infty} = 0$ This metric satisfies the **ultrametric inequality**: $d(a, c) \leq \max\{d(a, b), d(b, c)\}$, for all $a, b, c \in \mathbb{K}$. The ultrametric inequality has some strange consequences. For one thing, all triangles are isosceles - a triangle can have a shortest side but not a longest one. Also, any two closed balls are either disjoint or one is a subset of the other. Furthermore, every element of a ball is a center of the same ball. That is, $B(a, r) = B(c, r)$ whenever $c \in B(a, r)$, where the notation B denotes either open or closed balls.

The topology defined on $\mathbb{K}$ from its valuation metric is called the **valuation topology**. We can also define the valuation topology directly from v;

this is the topology with basic open sets: $\{ x \in \mathbb{K} \mid v(x-a) > n \}$, where $a \in \mathbb{K}$, $n \in \mathbb{N}$.

The following facts are straightforward:

Proposition 2.1 (Valuation and Topology) *Let* $(\mathbb{K}, v)$ *be a nontrivial valuation field. Then:*

(a) The valuation topology and the order topology on $\mathbb{K}$ *are the same.*

(b) The functions v *and* d *are continuous.*

(c) The notions of fundamental sequences, convergent sequences, and sequential completeness with respect to the valuation metric coincide with those notions with respect to absolute value.

(d) For any infinite sequence (a_n) *of elements of* $\mathbb{K}$*,* $a_n \to 0$ *if and only if* $v(a_n) \to \infty$*.*

Definition 2.3 (Spherically Complete) *A metric space is called* **spherically complete** *if every nested sequence of closed balls has nonempty intersection.*

A spherically complete metric space must be sequentially complete. Also, if a *metric space, generated by a valuation*, is spherically complete, then every collection of closed balls with the F.I.P. has nonempty intersection.

In contrast to Proposition 2.1(c) above, the following cannot be reversed:

Theorem 2.1 *If* $\mathbb{K}$ *is Cantor complete, then* $\mathbb{K}$ *is spherically complete with respect to any valuation on it.*

Proof: Assume $\mathbb{K}$ is Cantor complete and v is a valuation on $\mathbb{K}$. Let (B_n) be a strictly decreasing sequence of closed balls in $\mathbb{K}$. Our goal will be to define, for each n, a closed interval I_n such that $B_{n+1} \subseteq I_n \subseteq B_n$. This will make (I_n) a decreasing sequence of closed intervals, so by Cantor completeness, $\cap I_n \neq \emptyset$. But any element of $\cap I_n$ is clearly also in $\cap B_n$, establishing that $\mathbb{K}$ is spherically complete.

It remains to define I_n. Say $B_n = B(a_n, r_n)$, the closed ball of radius r_n centered at a_n. We have $a_{n+1} \in B_n$, so by our earlier remarks, a_{n+1} is also a center of B_n. That is, $B_n = B(a_{n+1}, r_n)$. Since $B_{n+1} \subset B_n$, we can choose some $c \in B_n - B_{n+1}$. Thus, $r_{n+1} < d(a_{n+1}, c) \leq r_n$. Let $s = |a_{n+1} - c|$. If $c < a_{n+1}$, let $I_n = [c, a_{n+1} + s]$. If $c > a_{n+1}$, let $I_n = [a_{n+1} - s, c]$. Since the distance between two numbers depends only on their difference, we have $B_{n+1} \subseteq I_n \subseteq B_n$, as desired. ▲

Thus, for a valuation field, $\aleph_1$-saturation implies Cantor completeness implies spherical completeness implies sequential completeness. A simple counterexample to the first converse is $\mathbb{R}$. Counterexamples to the latter two converses are described below.

An important category of non-archimedean fields are **generalized power series fields** or **Hahn fields**, introduced in H. Hahn [5]. Let $\mathbb{K}$ be a field and G an ordered abelian group. (In this paper, G is usually a subgroup of $(\mathbb{R}, +)$). For any formal power series $f = \sum_{g \in G} a_g t^g$, where each $a_g \in \mathbb{K}$, the **support** of f is $\{\, g \in G \mid a_g \neq 0 \,\}$. Then the set of all such f's whose support is well-ordered is a field (using ordinary polynomial-like addition and multiplication) denoted by $\mathbb{K}(t^G)$ or $\mathbb{K}((G))$. $\mathbb{K}$ is naturally imbedded in $\mathbb{K}(t^G)$ by mapping any a in $\mathbb{K}$ to at^0.

$\mathbb{K}(t^G)$ has a canonical G-valued Krull valuation in which each nonzero power series is mapped to the least exponent in its support. If $\mathbb{K}$ is ordered, then $\mathbb{K}(t^G)$ has a natural ordering in which an element is positive if and only if the coefficient corresponding to the least element in its support is positive. This ordering is compatible with the canonical valuation, and is the unique ordering on $\mathbb{K}(t^G)$ in which every positive power of t is between 0 and every positive element of $\mathbb{K}$.

One field of this type that has been studied extensively is $\mathbb{R}(t^{\mathbb{Z}})$, which is called the field of **Laurent series**. Another is the subfield $\mathbb{R}\langle t^{\mathbb{R}} \rangle$ of $\mathbb{R}(t^{\mathbb{Z}})$ consisting of those series whose support is either a finite set or an unbounded set of order type ω. (We will describe this field in a more concrete way in Section 3). This field was introduced by Levi-Civita in [10] and later was investigated by D. Laugwitz in [9] as a potential framework for the rigorous foundation of infinitesimal calculus before the advent of Robinson's non-standard analysis. It is also an example of a real-closed valuation field that is sequentially complete but not spherically complete (see V. Pestov [20], p. 67).

From W. Krull [8] and Theorem 2.12 of W.A.J. Luxemburg [14], it is known that every Hahn field of the form $\mathbb{K}(t^{\mathbb{R}})$ is spherically complete in its canonical valuation. In particular, $\mathbb{Q}(t^{\mathbb{R}})$ is spherically complete. But $\mathbb{Q}(t^{\mathbb{R}})$ is not Cantor complete, for the same reason that $\mathbb{Q}$ is not Cantor complete.

3 Asymptotic and Logarithmic Fields

Throughout this paper, we let ${}^*\mathbb{R}$ be a non-standard extension of $\mathbb{R}$ (Example 2.1). Every standard set, relation and function X has a nonstandard extension ${}^*X$; we may omit the symbol $*$, when no ambiguity arises from doing so. Until Section 5, we only need to assume that ${}^*\mathbb{R}$ is $\aleph_1$-saturated.

Let $\rho \in {}^*\mathbb{R}$ be a fixed positive infinitesimal. Following A. Robinson [23] we define the field of **Robinson's real ρ-asymptotic numbers** as the factor space ${}^\rho\mathbb{R} = \mathcal{M}_\rho({}^*\mathbb{R})/\mathcal{N}_\rho({}^*\mathbb{R})$, where

$$\mathcal{M}_\rho({}^*\mathbb{R}) = \{x \in {}^*\mathbb{R} \mid |x| \leq \rho^{-n} \text{ for some } n \in \mathbb{N}\},$$
$$\mathcal{N}_\rho({}^*\mathbb{R}) = \{x \in {}^*\mathbb{R} \mid |x| \leq \rho^{n} \text{ for all } n \in \mathbb{N}\},$$

are the sets of **ρ-moderate** and **ρ-null** numbers in ${}^*\mathbb{R}$, respectively. It is easy to check that $\mathcal{M}_\rho({}^*\mathbb{R})$ is a convex subring of ${}^*\mathbb{R}$ and $\mathcal{N}_\rho({}^*\mathbb{R})$ is a convex maximal ideal of $\mathcal{M}_\rho({}^*\mathbb{R})$. Therefore, ${}^\rho\mathbb{R}$ is a totally ordered field. The quotient mapping $q : \mathcal{M}_\rho({}^*\mathbb{R}) \to {}^\rho\mathbb{R}$, when restricted to $\mathbb{R}$, becomes an embedding of $\mathbb{R}$ into ${}^\rho\mathbb{R}$. The asymptotic number $q(\rho)$, denoted by s, or by $\overline{\rho}$, is called the **canonical scale** of ${}^\rho\mathbb{R}$.

It is simple to show that ${}^\rho\mathbb{R}$, like ${}^*\mathbb{R}$, is a real-closed, Cantor complete, non-archimedean extension of $\mathbb{R}$. The intervals $[-s^n, s^n]$, where $n \in \mathbb{N}$, form a base for the neighborhoods of 0 in the order topology on ${}^\rho\mathbb{R}$. So ${}^\rho\mathbb{R}$ is first countable, like $\mathbb{R}$ but unlike ${}^*\mathbb{R}$.

Let f be any function whose domain and range are subsets of $\mathbb{R}$. Chapter 4 of (A. Lightstone and A. Robinson [13]) describes the natural way of using ${}^*f$ to try to define a function ${}^\rho f$ that is an extension of f, and whose domain and range are subsets of ${}^\rho\mathbb{R}$. This procedure works for many but not all real functions. In particular, it works for all polynomials. It also works for the logarithm function, so we obtain a function ${}^\rho\ln : {}^\rho\mathbb{R}_+ \to {}^\rho\mathbb{R}$ satisfying ${}^\rho\ln(xy) = {}^\rho\ln(x) + {}^\rho\ln(y)$, etc. We shall write simply ln instead of ${}^\rho\ln$ when no confusion could arise. On the other hand, the exponential function cannot be defined in ${}^\rho\mathbb{R}$, since $e^{1/\rho} \notin \mathcal{M}_\rho({}^*\mathbb{R})$.

There is a canonical valuation v on ${}^\rho\mathbb{R}$ defined by $v(q(\alpha)) = \mathrm{st}(\log_\rho|\alpha|)$, where st is the standard part mapping in ${}^*\mathbb{R}$ and $\alpha \in \mathcal{M}_\rho({}^*\mathbb{R})$. It is easily shown that v is well-defined on ${}^\rho\mathbb{R}$ and is indeed a valuation (see A. Lightstone and A. Robinson [13], p. 79). In particular, we have $v(s) = 1$. Let $\mathcal{R}_v({}^\rho\mathbb{R}) = \{x \in {}^\rho\mathbb{R} \mid v(x) \geq 0\}$ and $\mathcal{I}_v({}^\rho\mathbb{R}) = \{x \in {}^\rho\mathbb{R} \mid v(x) > 0\}$ be the valuation ring and the valuation ideal of $({}^\rho\mathbb{R}, v)$, respectively, and $\widehat{{}^\rho\mathbb{R}} = \mathcal{R}_v({}^\rho\mathbb{R})/\mathcal{I}_v({}^\rho\mathbb{R})$ be

the residue class field of $({}^\rho\mathbb{R}, v)$ (called also *logarithmic field*). Notice that $\widehat{{}^\rho\mathbb{R}}$ can be viewed as well as a quotient field of a subring of ${}^*\mathbb{R}$. We let $\widehat{q} : \mathcal{R}_v({}^\rho\mathbb{R}) \to \widehat{{}^\rho\mathbb{R}}$ be the quotient map.

For more details about ${}^\rho\mathbb{R}$ we refer the reader to A. Robinson [23], A. H. Lightstone and A. Robinson [13], W.A.J. Luxemburg [14], and V. Pestov [20].

4 The Quasi-Standard Part Mapping

We always view ${}^\rho\mathbb{R}$ as an extension of $\mathbb{R}$.

Definition 4.1 (Feasible): *Let $B \subseteq {}^\rho\mathbb{R}$. We will say that B is* ***feasible*** *if $\mathbb{R}(B) \subseteq \mathcal{R}_v({}^\rho\mathbb{R})$, where $\mathbb{R}(B)$ denotes the subfield of ${}^\rho\mathbb{R}$ generated by $\mathbb{R} \cup B$.*

The following proposition is immediate.

Proposition 4.1 *(a) For any $B \subseteq {}^\rho\mathbb{R}$, B is feasible if and only if every non-zero element of $\mathbb{R}(B)$ has valuation zero.*

(b) Suppose that $B \subseteq {}^\rho\mathbb{R}$ and, for any nonzero polynomial $P \in \mathbb{R}[t_1, t_2, ..., t_n]$ and any distinct $b_1, b_2, ..., b_n \in B$, $v(P(b_1, b_2, ..., b_n)) = 0$. Then B is feasible.

Theorem 4.1 (Existence): *Let B be feasible. Then:*

(a) There is a maximal field $\widehat{\mathbb{R}}$, which is a a subring of $\mathcal{R}_v({}^\rho\mathbb{R})$ that contains $\mathbb{R}(B)$, in symbols, $\mathbb{R}(B) \subset \widehat{\mathbb{R}} \subset \mathcal{R}_v({}^\rho\mathbb{R})$

(b) Any such field $\widehat{\mathbb{R}}$ is real-closed.

(c) Any such field $\widehat{\mathbb{R}}$ is isomorphic to $\widehat{{}^\rho\mathbb{R}}$ via the quotient mapping $\widehat{q}$.

Proof: Results similar to this appear in various places in the literature, such as Lemma 2 of MacLane [15]. We give a simple independent proof.

(a) Given B, we obtain the desired field by applying Zorn's Lemma to the collection of all fields $\mathbb{K}$ such that $\mathbb{R}(B) \subseteq \mathbb{K} \subseteq \mathcal{R}_v({}^\rho\mathbb{R})$, to obtain a maximal such field.

(b) Given a maximal field $\mathbb{K}$ as in (a), Let $\mathrm{cl}(\mathbb{K})$ be the relative algebraic closure of $\mathbb{K}$ in ${}^\rho\mathbb{R}$. Since ${}^\rho\mathbb{R}$ is real-closed, so is $\mathrm{cl}(\mathbb{K})$. So it suffices to show that $\mathrm{cl}(\mathbb{K}) \subset \mathcal{R}_v({}^\rho\mathbb{R})$, since this implies that $\mathrm{cl}(\mathbb{K}) = \mathbb{K}$ by the maximality of $\mathbb{K}$. So let $x \in \mathrm{cl}(\mathbb{K})$. Then x is a root of some polynomial $t^n + a_{n-1}t^{n-1} + ... + a_0$, with $a_k \in \mathbb{K}$. But the bound $|x| \leq 1 + |a_0| + ... + |a_{n-1}|$ implies that $v(x) \geq 0$, so $x \in \mathcal{R}_v({}^\rho\mathbb{R})$.

(c) Since $\mathbb{K} \subseteq \mathcal{R}_v({}^\rho\mathbb{R})$, the quotient mapping $\widehat{q}$, restricted to $\mathbb{K}$, is a field homomorphism from $\mathbb{K}$ to $\widehat{{}^\rho\mathbb{R}}$. Furthermore, $\widehat{q}$ is one-to-one, because if $\widehat{q}(x) = \widehat{q}(y)$, then $v(x-y) > 0$, which implies that $x = y$ because v is trivial on $\mathbb{K}$. (I.e., $v(x) > 0$ for a nonzero $x \in \mathbb{K}$ would imply $v(1/x) < 0$, contradicting that $\mathbb{K} \subseteq \mathcal{R}_v({}^\rho\mathbb{R})$.) Thus $\widehat{q}$ defines an isomorphism between $\mathbb{K}$ and its image $\widehat{q}(\mathbb{K})$. It only remains to prove that $\widehat{q}(\mathbb{K}) = \widehat{{}^\rho\mathbb{R}}$. If not, then there's a $\xi \in \mathcal{R}_v({}^\rho\mathbb{R})$ such that $\widehat{q}(\xi) \notin \widehat{q}(\mathbb{K})$. Now let $P \in \mathbb{K}[t], P \neq 0$. We claim that $v(P(\xi)) = 0$. Otherwise, we would have $v(P(\xi)) > 0$, since $v(\xi) = 0$. But then $\widehat{q}(P(\xi)) = 0$, or equivalently, $\widehat{P}(\widehat{q}(\xi)) = 0$ in $\widehat{{}^\rho\mathbb{R}}$, where $\widehat{P}$ is the polynomial $\widehat{q}(P)$ in $\widehat{q}(\mathbb{K})[t]$. So $\widehat{q}(\xi)$ is algebraic over $\widehat{\mathbb{K}}$. But, since $\mathbb{K}$ is real-closed by (b), its isomorphic image $\widehat{q}(\mathbb{K})$ is a real-closed subfield of the ordered field $\widehat{{}^\rho\mathbb{R}}$. Thus $\widehat{q}(\xi) \in \widehat{q}(\mathbb{K})$, contradicting the assumption about ξ. So $v(P(\xi)) = 0$. It follows that $v(P(\xi)/Q(\xi)) = 0$ whenever $P, Q \in \mathbb{K}[t], P, Q \neq 0$. In other words, $\mathbb{K}(\xi) \subseteq \mathcal{R}_v({}^\rho\mathbb{R})$. But since $\xi \notin \mathbb{K}$, this contradicts the maximality of $\mathbb{K}$. ▲

Note that this theorem is not vacuous, since we can let $B = \varnothing$.

Let $\widehat{j} : \widehat{{}^\rho\mathbb{R}} \to {}^\rho\mathbb{R}$ be the inverse of the isomorphism $\widehat{q}$ described in Theorem 4.1. So $\widehat{j}$ is an isomorphic embedding of $\widehat{{}^\rho\mathbb{R}}$ in ${}^\rho\mathbb{R}$. We refer to j as a **B-invariant embedding** of $\widehat{{}^\rho\mathbb{R}}$ into ${}^\rho\mathbb{R}$. Of course, this embedding is not unique, even for fixed B.

A field $\widehat{\mathbb{R}}$ that satisfies Theorem 4.1 is sometimes called a **field of representatives** for $\mathcal{R}_v({}^\rho\mathbb{R})$. We will also call it a **B-copy of $\widehat{{}^\rho\mathbb{R}}$ in ${}^\rho\mathbb{R}$**. Henceforth, $\widehat{\mathbb{R}}$ always denotes such a field.

Corollary 4.1 *$\mathcal{R}_v({}^\rho\mathbb{R}) = \widehat{\mathbb{R}} \oplus \mathcal{I}_v({}^\rho\mathbb{R})$.*

Proof: Let $\xi \in \mathcal{R}_v({}^\rho\mathbb{R})$. By Theorem 4.1, we know there's an x in $\widehat{\mathbb{R}}$ such that $\widehat{q}(\xi) = \widehat{q}(x)$. In other words, $\xi = x + h$, for some $h \in \mathcal{I}_v({}^\rho\mathbb{R})$. And since $v|\widehat{\mathbb{R}}$ is trivial, this representation $\xi = x + h$ is unique. ▲

Definition 4.2 *Given $\widehat{\mathbb{R}}$, we define the* ***quasi-standard part mapping*** *$\widehat{\mathrm{st}} : \mathcal{R}_v({}^\rho\mathbb{R}) \to \widehat{\mathbb{R}}$ by $\widehat{\mathrm{st}}(x+h) = x$, for any $x \in \widehat{\mathbb{R}}$ and $h \in \mathcal{I}_v({}^\rho\mathbb{R})$.*

The following properties of $\widehat{\mathrm{st}}$ are straightforward:

Proposition 4.2 (Properties) *(a) $\widehat{\mathrm{st}}$ is an ordered ring homomorphism from $\mathcal{R}_v({}^\rho\mathbb{R})$ onto $\widehat{{}^\rho\mathbb{R}}$.*

(b) $\widehat{\mathrm{st}}|\widehat{\mathbb{R}}$ *is the identity. In particular, all real numbers and all asymptotic numbers in* B *are fixed points of* $\widehat{\mathrm{st}}$.

(c) $\widehat{\mathrm{st}} \circ \widehat{\mathrm{st}} = \widehat{\mathrm{st}}$.

(d) The restriction of $\widehat{\mathrm{st}}$ *to* $\mathbb{R} \oplus \mathcal{I}_v({}^\rho\mathbb{R})$ *coincides with* st, *the usual standard part mapping in* ${}^\rho\mathbb{R}$ *(hence the term "quasi-standard part mapping").*

(e) $\widehat{\mathrm{st}} = \widehat{j} \circ \widehat{q}$.

For applications in asymptotic analysis involving Laurent asymptotic expansions or, more generally, expansions in Levi-Civita series (with real or complex coefficients), it usually suffices to apply Theorem 4.1 with $B = \varnothing$. For other applications, it can be fruitful to use a non-empty B, in order to provide the mapping $\widehat{\mathrm{st}}$ with particular fixed points other than the reals. We conclude this section by discussing a choice of B that is useful for applications involving asymptotic expansions with logarithm terms:

Notation 4.1 (Multiple Logarithms): *(a) Let* $\lambda_1 = |\ln\rho|$ *and, inductively,* $\lambda_{n+1} = \ln(\lambda_n)$. *Notice that the* λ_n*'s form a decreasing sequence of infinitely large positive numbers in* ${}^*\mathbb{R}$.

(b) Let $l_n = q(\lambda_n)$, *for* $n \in \mathbb{N}$. *So the* l_n*'s form a decreasing sequence of infinitely large positive numbers in* ${}^\rho\mathbb{R}$. *However,* $v(l_n) = 0$.

Another way to define the l_n's is to use the function ${}^\rho\ln$ in ${}^\rho\mathbb{R}$, as discussed near the end of Section 1. Specifically, $l_1 = {}^\rho\ln(1/s)$, and $l_{n+1} = {}^\rho\ln(l_n)$.

Lemma 4.1 *The set* $\mathcal{L} = \{l_n : n \in \mathbb{R}\}$ *is feasible. So are the sets* $\mathcal{S} = \{e^{\pm i\pi/s} \ln^n s : n = 0, 1, 2, \dots\}$ *and* $\mathcal{L} \cup \mathcal{S}$.

Proof: By Proposition 4.1, it suffices to prove that if $P \in \mathbb{R}[t_1, t_2, ..., t_n]$ and $P \neq 0$, then $v(P(l_1, l_2, ..., l_n)) = 0$. First note that this is obviously true when P is a monomial, since each l_n has valuation 0. If P has more than one term, we just need to show that $P(l_1, l_2, ..., l_n)$ has a "dominant term," compared to which all other terms are infinitesimal. (Technically, this is the greatest term in the polynomial with respect to the lexicographic ordering on the sequence of its exponents.) For this it suffices to show that $m > k$ implies $(\lambda_m)^p/(\lambda_k)^q \approx 0$ whenever p and q are in $\mathbb{R}$. By the transfer principle, this is equivalent to a limit statement in standard mathematics, which is easily verified by l'Hopital's Rule. The sets S and $\mathcal{L} \cup \mathcal{S}$ are treated similarly. ▲

If we apply Theorem 4.1 with $B = \mathcal{L} \cup \mathcal{S}$, the resulting field of representatives $\widehat{\mathbb{R}}$ contains all the l_n's and $e^{\pm i\pi/s} \ln^n s$, which are fixed points of $\widehat{\mathrm{st}}$. Under that assumption we have $\widehat{\mathrm{st}}(\ln s + r + s) = \ln s + r$, $\widehat{\mathrm{st}}(e^{\pm i\pi/s} + r + s^2) = e^{\pm i\pi/s} + r$, $\widehat{\mathrm{st}}(s) = 0$ where $r \in \mathbb{R}$.

5 The Imbedding of $\widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$ in ${}^\rho\mathbb{R}$

Recall the definition in Section 1 of the field $\mathbb{R}\langle t^{\mathbb{R}}\rangle$ We can similarly define $\mathbb{K}\langle t^{\mathbb{R}}\rangle$ for any field $\mathbb{K}$, and we will refer to all fields of this type as **Levi-Civita fields**. A. Robinson [23] showed that $\mathbb{R}\langle t^{\mathbb{R}}\rangle$ can be embedded in ${}^\rho\mathbb{R}$. We now generalize Robinson's result (and Theorem 4.1 above) by providing an imbedding of $\widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$ in ${}^\rho\mathbb{R}$. Since $\widehat{{}^\rho\mathbb{R}}$ is isomorphic to a subfield $\widehat{\mathbb{R}}$ of ${}^\rho\mathbb{R}$, we temporarily identify $\widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$ with $\widehat{\mathbb{R}}\langle t^{\mathbb{R}}\rangle$

For uniformity, we assume that any series $A \in \widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$ is written in the form $A = \sum_{k=0}^{\infty} a_k t^{r_k}$, where the sequence (r_k) of reals is strictly increasing and unbounded, $a_k \in \widehat{{}^\rho\mathbb{R}}$ (or $\widehat{\mathbb{R}}$), and if any $a_k = 0$, then $a_m = 0$ for every $m > k$. Naturally, two such series are considered equal if any term that is in one series but not the other has coefficient 0. If the series $\sum_{k=0}^{\infty} a_k\, x^{r_k}$ is convergent in ${}^\rho\mathbb{R}$ for some $x \in {}^\rho\mathbb{R}$, we denote its sum by $A(x)$ and write $A(x) = \sum_{k=0}^{\infty} a_k\, x^{r_k}$. Let $\mathcal{I}_v({}^\rho\mathbb{R}_+) = \{h \in {}^\rho\mathbb{R}_+ \mid v(h) > 0\}$ denote the multiplicative group of scales in ${}^\rho\mathbb{R}$, where ${}^\rho\mathbb{R}_+$ is the set of positive elements of ${}^\rho\mathbb{R}$.

Lemma 5.1 *Let $A \in \widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$, $h \in \mathcal{I}_v({}^\rho\mathbb{R}_+)$. Then*

(a) The series $A(h) = \sum_{k=0}^{\infty} a_k\, h^{r_k}$ is convergent in ${}^\rho\mathbb{R}$.

(b) If $A(h) \neq 0$, then $v(A(h)) = r_0 v(h)$.

Proof: (a) If $A(h)$ is a finite series, there's nothing to prove. Otherwise, since $v(a_k) \geq 0$ and $v(h) > 0$, it follows that $v(a_k h^{r_k}) \to \infty$ as $k \to \infty$. By remarks made in Section 1, this implies that $a_k h^{r_k} \to 0$, which in turn implies that $A(h)$ converges.

(b) This follows from the fact that $a_0 h^{r_0}$ is the dominant term in any partial sum of $A(h)$, and the continuity of v. ▲

Theorem 5.1 (The Imbedding) *For any fixed $h \in \mathcal{I}_v({}^\rho\mathbb{R}_+)$, the function $M_h : \widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle \to {}^\rho\mathbb{R}$ defined by $M_h(A) = A(h)$ is a continuous ordered field embedding.*

Proof: The previous lemma showed that M_h is defined on $\widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$. The fact that M_h is a ring homomorphism is clear. We also need to show that M_h is order-preserving: for any $A \in \widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$, we have $A > 0$ iff $a_0 > 0$ iff $A(h) > 0$, since $a_0 h^{r_0}$ is the dominant term of $A(h)$. So M_h is an ordered ring embedding. To show that M_h is continuos, note that $v(M_h(A)) = v(A(h)) = r_0 v(h)$

by the previous lemma, which equals $V(A)v(h)$, where V is the canonical valuation on $\widehat{^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$. So M_h is valuation-preserving except for a factor of $v(h)$. This immediately implies continuity of M_h. ▲

Note that the embedding M_h depends on the choice of $\widehat{\mathbb{R}}$ as well as h (not to mention ρ). The particular embedding M_s, where $s = \overline{\rho}$ is the canonical scale of $^\rho\mathbb{R}$, is an extension of the embedding Φ, defined in A. Robinson [23]. Unlike a typical M_h, M_s is also valuation-preserving.

Corollary 5.1 *Let $\widehat{^\rho\mathbb{R}}\langle h^{\mathbb{R}}\rangle$ denote the range of M_h. The inverse of the mapping M_h is given explicitly as follows: for any y in $\widehat{^\rho\mathbb{R}}\langle h^{\mathbb{R}}\rangle$, $y = M_h(A) = A(h)$, where the series $A = \sum_{k=0}^{\infty} a_k\, t^{r_k}$ is defined inductively by:*

$$r_n = v\left(y - \sum_{k<n} a_k h^{r_k}\right)/v(h), \quad a_n = \widehat{\mathrm{st}}\left((y - \sum_{k<n} a_k h^{r_k})/h^{r_n}\right),$$

with the understanding that if this algorithm ever gives $r_n = \infty$, the series terminates at the previous term.

Proof: We verify the algorithm for r_0 and a_0. Its validity for subsequent terms follows by the same reasoning. First of all, if $y = 0$, then the algorithm gives $r_0 = \infty$, whence A is the "null series," which is the zero element of $\widehat{^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$. If $y \neq 0$, we know from Lemma 5.1(b) that $v(y) = r_0 v(h)$, so $r_0 = v(y)/v(h)$, as desired. It then follows that $v(y/h^{r_0}) = v(y) - r_0 v(h) = 0$, so $y/h^{r_0} \in \mathcal{R}_v(^\rho\mathbb{R})$. Therefore, $\widehat{\mathrm{st}}(y/h^{r_0})$ is well-defined. Furthermore, by definition of $\widehat{\mathrm{st}}$, the value of a_0 defined by our algorithm is the unique member of $\widehat{^\rho\mathbb{R}}$ (more precisely, of the field of representatives $\widehat{\mathbb{R}}$) such that $v(y/h^{r_0} - a_0) > 0$. But the correct a_0 must satisfy $v(y - a_0 h^{r_0}) = v(a_1 h^{r_1}) = r_1 v(h) > r_0 v(h))$, or, equivalently, $v(y/h^{r_0} - a_0) > 0$, so the defined value of a_0 is correct. ▲

6 The Hahn Field Representation of $^\rho\mathbb{R}$

In this section we prove the existence of valuation-preserving isomorphisms between $^\rho\mathbb{R}$ and the Hahn field $\widehat{^\rho\mathbb{R}}(t^{\mathbb{R}})$. A similar result in the setting of ultraproducts appears in B. Diarra [3] (Corollary to Proposition 8). Our result may be more general in a couple of ways. In particular, our isomorphisms allows us to specify fixed points, as discussed at the end of Section 2.

Unless stated otherwise, the results in this section pertain to general Krull's valuations, not necessarily convex or real-valued (W. Krull [8]).

Definition 6.1 (Maximal Extensions): *Let $(\mathbb{K}_i, v_i)$ be valuation fields, $i = 1, 2$. If $\mathbb{K}_1$ is a subfield of $\mathbb{K}_2$ and $v_2|\mathbb{K}_1 = v_1$, we say that $(\mathbb{K}_2, v_2)$ is an* **extension** *of $(\mathbb{K}_1, v_1)$. If, in addition, the value groups G_{v_1} and G_{v_2} are equal and the residue class fields $\widehat{\mathbb{K}}_1$ and $\widehat{\mathbb{K}}_2$ coincide (in the sense that the natural embedding of $\widehat{\mathbb{K}}_1$ in $\widehat{\mathbb{K}}_2$ is surjective), the extension is called* ***immediate***. *A valuation field with no proper immediate extensions is called* ***maximal***.

W.A.J. Luxemburg [14] proved that maximality is equivalent to spherical completeness in fields with convex, real-valued valuations. W. Krull [8] proved that every valuation field has a maximal immediate extension. We are more interested in the following result, due to Kaplansky [7] (characteristic 0 case of Theorem 5):

Theorem 6.1 (Kaplansky): *Let $(\mathbb{K}, v)$ be a valuation field whose residue class field has characteristic 0. Then the maximal immediate extension of $(\mathbb{K}, v)$ is unique, in the sense that between any two maximal immediate extensions of $(\mathbb{K}, v)$ there is a valuation-preserving isomorphism that is the identity on $\mathbb{K}$.*

Corollary 6.1 (Extension): *Let $(\mathbb{K}_i, v_i)$, for $i = 1, 2$, be a valuation field whose residue class field has characteristic 0, and let $(\mathbb{L}_i, w_i)$ be a maximal immediate extension of $(\mathbb{K}_i, v_i)$. Then any valuation-preserving isomorphism between $\mathbb{K}_1$ and $\mathbb{K}_2$ can be extended to such an isomorphism between $\mathbb{L}_1$ and $\mathbb{L}_2$.*

Theorem 6.2 (Isomorphism): *For any choice of the field of representatives $\widehat{\mathbb{R}}$ of $\widehat{{}^\rho\mathbb{R}}$, there is a valuation-preserving isomorphism $J : {}^\rho\mathbb{R} \to \widehat{{}^\rho\mathbb{R}}(t^{\mathbb{R}})$ with the following additional properties:*

(a) J^{-1} restricted to $\widehat{{}^\rho\mathbb{R}}\langle t^{\mathbb{R}}\rangle$ is the mapping M_s defined in Section 5.

(b) J maps the scale s of ${}^\rho\mathbb{R}$ to the indeterminate t of $\widehat{{}^\rho\mathbb{R}}(t^{\mathbb{R}})$.

(c) J restricted to $\widehat{{}^\rho\mathbb{R}}$ is the quotient mapping $\widehat{q}$.

Proof: We apply Corollary 6.1 with $\mathbb{K}_1 = \widehat{{}^\rho\mathbb{R}}\langle s^{\mathbb{R}}\rangle$, $\mathbb{K}_2 = \widehat{{}^\rho\mathbb{R}}(t^{\mathbb{R}})$, $\mathbb{L}_1 = {}^\rho\mathbb{R}$, and $\mathbb{L}_2 = \widehat{{}^\rho\mathbb{R}}(t^{\mathbb{R}})$ (all with their usual valuation), and the valuation-preserving isomorphism $(M_s)^{-1}$ between $\mathbb{K}_1$ and $\mathbb{K}_2$ (see Section 5). $\mathbb{L}_1$ is an immediate

extension of $\mathbb{K}_1$ because both have valuation group $\mathbb{R}$ and residue class field $\widehat{{}^\rho\mathbb{R}}$ (and the residue class fields coincide in the sense described in the definition of immediateness). Similarly, $\mathbb{L}_1$ is an immediate extension of $\mathbb{K}_1$. The maximality of $({}^\rho\mathbb{R}, v)$ is proved in W.A.J. Luxemburg [14]. The maximality of Hahn fields is proved in W. Krull [8]. Thus, by Corollary 6.1, there is a valuation-preserving isomorphism J between ${}^\rho\mathbb{R}$ and $\widehat{{}^\rho\mathbb{R}}(t^{\mathbb{R}})$ that satisfies (a). Properties (b) and (c) follow directly from the definition of M_s. ▲

Remark 6.1 *Note that the isomorphism provided by Theorem 6.2 is nonunique and nonconstructive at several stages. First, we have some freedom in the choice of the set B. Then, the definition of ${}^\rho\mathbb{R}$ (Theorem 4.1) uses Zorn's Lemma. Finally, the definition of the isomorphism in Kaplansky's result (Theorem 6.1) is also nonconstructive. We could also use any M_h instead of M_s as the basis for J if we don't need J to be valuation-preserving.*

Remark 6.2 *Theorem 6.2 and most of our other results so far about ${}^\rho\mathbb{R}$ hold for any spherically complete, real-closed extension of $\mathbb{R}$ with a convex, real-valued nonarchimedean valuation.*

7 Isomorphism Between ${}^*\mathbb{R}$ and $\widehat{{}^\rho\mathbb{R}}$

Given ${}^*\mathbb{R}$ one might ask if there is essentially just one field ${}^\rho\mathbb{R}$. In other words, does the structure of ${}^\rho\mathbb{R}$ depend on the choice of ρ? This question leads to two cases. For positive infinitesimals ρ_1 and ρ_2, suppose that $\ln\rho_1/\ln\rho_2$ is finite but not infinitesimal. This condition is easily seen to be equivalent to $\mathcal{M}_{\rho_1}({}^*\mathbb{R}) = \mathcal{M}_{\rho_2}({}^*\mathbb{R})$. Therefore, since the maximal ideal $\mathcal{N}_\rho({}^*\mathbb{R})$ is determined uniquely by $\mathcal{M}_\rho({}^*\mathbb{R})$, this condition implies that ${}^{\rho_1}\mathbb{R}$ and ${}^{\rho_2}\mathbb{R}$ are not just isomorphic but identical. Without assuming this condition, we can still prove that any two ${}^\rho\mathbb{R}$'s defined from a particular ${}^*\mathbb{R}$ are isomorphic, but only under the assumption that ${}^*\mathbb{R}$ is a special model (defined later in this section). In fact, under this assumption, $\widehat{{}^\rho\mathbb{R}}$ (for any ρ) is actually isomorphic to ${}^*\mathbb{R}$. This is perhaps surprising, since it seems that ${}^\rho\mathbb{R}$ is a "much smaller" field than ${}^*\mathbb{R}$, and $\widehat{{}^\rho\mathbb{R}}$ in turn is "much smaller" than ${}^\rho\mathbb{R}$.

${}^*\mathbb{R}$ is called **fully saturated** if it is card(${}^*\mathbb{R}$)-saturated. We will first prove our results for fully saturated ${}^*\mathbb{R}$, and then generalize them to special ${}^*\mathbb{R}$.

We briefly review some basic concepts of model theory: Let $\mathcal{L}$ be a first-order language. (For our purposes, $\mathcal{L}$ is the first-order language of an ordered

ring, with two binary operation symbols and one binary relation symbol besides equality). Two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are called **elementarily equivalent**, written $\mathcal{A} \equiv \mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ satisfy exactly the same statements of $\mathcal{L}$ (with no free variables). Also, $\mathcal{A}$ is said to be **elementarily embedded** in $\mathcal{B}$, written $\mathcal{A} \prec \mathcal{B}$, if $\mathcal{A}$ is a substructure of $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ satisfy exactly the same formulas of $\mathcal{L}$ with free variables interpreted as any members of the domain of $\mathcal{A}$.

The second condition in the definition of $\mathcal{A} \prec \mathcal{B}$ is much stronger than $\mathcal{A} \equiv \mathcal{B}$. For further details on these concepts, see Chang and Keisler [1], which also includes the following important results:

Theorem 7.1 (Model Theory of Fields) *(a)* Tarski Theorem*: All real-closed fields are elementarily equivalent (Theorem 5.4.4 in [1]).*

(b) Model-Completeness: *If $\mathcal{B}$ is a real-closed field, then $\mathcal{A} \prec \mathcal{B}$ if and only if $\mathcal{A}$ is a real-closed subfield of $\mathcal{B}$ (page 110 in [1]).*

(c) Elementary Chain Theorem: *Let $\{ \mathcal{A}_i \mid i \in I \}$ be a family of structures such that I is well-ordered, and $\mathcal{A}_i \prec \mathcal{A}_j$ whenever $i, j \in I$ and $i < j$. Then $\mathcal{A}_i \prec \cup_{i \in I} \mathcal{A}_i$, for every $i \in I$ (Theorem 3.1.13 in [1]).*

(d) Uniqueness of special and fully saturated models: *Any two elementarily equivalent, special models of the same cardinality are isomorphic (and every fully saturated model is special). (Theorems 5.1.17 and 5.1.13. in [1])*

Theorem 7.2 *If ${}^*\mathbb{R}$ is fully saturated, then ${}^*\mathbb{R}$ is isomorphic to any residue class field $\widehat{{}^\rho\mathbb{R}}$ as defined in Section 3.*

Proof: Assume ${}^*\mathbb{R}$ is fully saturated, and let $\kappa = \mathrm{card}({}^*\mathbb{R})$. We will use Theorem 7.1 (d). ${}^*\mathbb{R}$ is real-closed by the transfer principle, and $\widehat{{}^\rho\mathbb{R}}$ is real-closed by Theorem 4.1, (b) and (c). So, by Theorem 7.1 (a), ${}^*\mathbb{R} \equiv {}^\rho\mathbb{R}$. Recall that ${}^\rho\mathbb{R}$ is a quotient field of a subring of ${}^*\mathbb{R}$, and $\widehat{{}^\rho\mathbb{R}}$ is defined similarly from ${}^\rho\mathbb{R}$. We now combine these operations: let Q be the composition $\widehat{q} \circ q$. The domain of Q is

$$\mathcal{F}_\rho({}^*\mathbb{R}) = \{x \in {}^*\mathbb{R} \mid |x| < 1/\sqrt[n]{\rho} \text{ for all } n \in \mathbb{N}\}.$$

The kernel of Q is

$$\mathcal{I}_\rho({}^*\mathbb{R}) = \{x \in^* \mathbb{R} \mid |x| < \sqrt[n]{\rho} \text{ for some } n \in \mathbb{N}\}.$$

It is clear from the above that $\mathrm{card}(\widehat{{}^\rho\mathbb{R}}) \leq \kappa$. Therefore, if we can show that $\widehat{{}^\rho\mathbb{R}}$ is κ-saturated, we are done because $\widehat{{}^\rho\mathbb{R}}$ is then fully saturated and has

the same cardinality as ${}^*\mathbb{R}$. (Here we are using the simple fact that a κ-saturated model must have cardinality at least κ.) So let

$$\mathcal{C} = \{(a_i, b_i) | i \in I\},$$

be a collection of fewer than κ open intervals in $\widehat{{}^\rho\mathbb{R}}$ with the F.I.P. Let $A = \{a_i\}$ and $B = \{b_i\}$. For each $i \in I$, let $\tilde{a}_i \in {}^*\mathbb{R}$ be such that $Q(\tilde{a}_i) = a_i$. Similarly let $Q(\tilde{b_i}) = b_i$ for every $i \in I$. Our goal is to prove that $\mathcal{C}$ has nonempty intersection. We proceed by cases. If A has a greatest element and B has a least element, this follows trivially. If A has no greatest element and B has no least element, let $\tilde{\mathcal{C}} = \{(\tilde{a}_i, \tilde{b}_i) | i \in I\}$. Since $\mathcal{C}$ has the FIP, so does $\tilde{\mathcal{C}}$. Since ${}^*\mathbb{R}$ is fully saturated, there is some $x \in \bigcap \tilde{\mathcal{C}}$. It is then easily seen that $Q(x) \in \bigcap \mathcal{C}$.Finally, we consider the case that A has a greatest element a, but B has no least element. (The reverse situation is handled similarly). Now let $\tilde{a}$ be such that $Q(\tilde{a}) = a$, and define $\tilde{\mathcal{C}} = \{(\tilde{a}+\rho^{1/n}, \tilde{b}_i) | i \in I, n \in \mathbb{N}\}$. $\tilde{\mathcal{C}}$ has cardinality less than κ and has the F.I.P., so $\bigcap \tilde{\mathcal{C}} \neq \varnothing$. Let $x \in \bigcap \tilde{\mathcal{C}}$. Clearly, $Q(x) < b_i$ for every $i \in I$. And since $x > \tilde{a} + \rho^{1/n}$ for every n, $Q(x) > Q(\tilde{a}) = a$. Thus $Q(x) \in \bigcap \mathcal{C}$. ▲

The following corollary is immediate.

Corollary 7.1 *Let ρ_1 and ρ_2 be positive infinitesimals in the same fully saturated ${}^*\mathbb{R}$. Then:*
(a) The residue class fields $\widehat{{}^{\rho_1}\mathbb{R}}$ and $\widehat{{}^{\rho_2}\mathbb{R}}$ are isomorphic.
(b) The fields ${}^{\rho_1}\mathbb{R}$ and ${}^{\rho_2}\mathbb{R}$ are isomorphic.
(c) In fact, any ${}^\rho\mathbb{R}$ is isomorphic to the Hahn field ${}^*\mathbb{R}(t^{\mathbb{R}})$.

Remark 7.1 *Parts (a) and (b) of this corollary appear on page 196 of W.A.J. Luxemburg [14], without proof and without stating the necessity of any saturation assumption beyond $\aleph_1$-saturation.*

Since we are viewing ${}^*\mathbb{R}$, ${}^\rho\mathbb{R}$, and $\widehat{{}^\rho\mathbb{R}}$ as extension fields of $\mathbb{R}$, it is desirable to know when these isomorphisms can be specified to be "over $\mathbb{R}$":

Corollary 7.2 *Assume that ${}^*\mathbb{R}$ is fully saturated, and* $\text{card}({}^*\mathbb{R}) > \text{card}(\mathbb{R})$. *Then the isomorphisms described in Theorem 7.2 and Corollary 7.1 can be chosen to be the identity on $\mathbb{R}$.*

Proof: We know that $\mathbb{R}$ is a real-closed subfield of both the real-closed fields ${}^*\mathbb{R}$ and ${}^\rho\mathbb{R}$. So, by Theorem 7.1 (b), $\mathbb{R} \prec {}^*\mathbb{R}$ and $\mathbb{R} \prec \widehat{{}^\rho\mathbb{R}}$. These conditions

are clearly equivalent to $\mathbb{R}' \equiv {}^*\mathbb{R}'$ and $\mathbb{R}' \equiv \widehat{{}^\rho\mathbb{R}}'$, where these three new structures are obtained by adding individual constants for every real number to $\mathbb{R}$, ${}^*\mathbb{R}$, and ${}^\rho\mathbb{R}$, respectively. (These new structures are $\mathcal{L}'$-structures, where $\mathcal{L}'$ is the first-order language of an ordered ring, augmented by a constant symbol for each real number.) Thus, ${}^*\mathbb{R}' \equiv {}^\rho\mathbb{R}'$. Also, since ${}^*\mathbb{R}$ and $\widehat{{}^\rho\mathbb{R}}$ are $\mathrm{card}(\mathbb{R})^+$-saturated, augmenting them by adding $\mathrm{card}(\mathbb{R})$ constants does not decrease their saturation (C.C. Chang and H.J. Keisler [1], Proposition 5.1.1(iii)). In other words, ${}^*\mathbb{R}'$ and ${}^\rho\mathbb{R}'$ are also fully saturated. Therefore, by Theorem 7.1 (d), ${}^*\mathbb{R}'$ and $\widehat{{}^\rho\mathbb{R}}'$ are isomorphic. But an isomorphism between ${}^*\mathbb{R}'$ and ${}^\rho\mathbb{R}'$ is simply an isomorphism between ${}^*\mathbb{R}$ and $\widehat{{}^\rho\mathbb{R}}$ that is the identity on $\mathbb{R}$. ▲

It is well known that fully saturated models are hard to come by. To prove the existence of a fully saturated ${}^*\mathbb{R}$ requires assuming either some form of the continuum hypothesis, or that $\mathrm{card}({}^*\mathbb{R})$ is inaccessible. For example, if ${}^*\mathbb{R}$ is a countable ultrapower of $\mathbb{R}$, then ${}^*\mathbb{R}$ is $\aleph_1$-saturated and $\mathrm{card}({}^*\mathbb{R}) = 2^{\aleph_0}$. So the ordinary continuum hypothesis would make this ${}^*\mathbb{R}$ fully saturated. But without a special assumption such as the continuum hypothesis, we cannot prove that Theorem 7.2 and Corollaries 7.1 and 7.2 are non-vacuous. We will now fix this problem by considering special models.

Definition 7.1 *Let $\mathcal{A}$ be a structure with domain A. $\mathcal{A}$ is called **special** if $\mathcal{A}$ is the union of a chain of structures $\{\mathcal{A}_\kappa\}$, where κ ranges over all cardinals less than* Card*(A), each $\mathcal{A}_\kappa$ is κ^+-saturated, and $\mathcal{A}_{\kappa_1} \prec \mathcal{A}_{\kappa_2}$ whenever $\kappa_1 < \kappa_2$. The family $\{\mathcal{A}_\kappa\}$ is called a* **specializing chain** *for $\mathcal{A}$.*

For structures whose cardinality is regular, specialness is equivalent to full saturation. However, while fully saturated models of singular cardinality cannot exist, the existence of a special ${}^*\mathbb{R}$ (of an appropriate singular cardinality) can be proved in ZFC (see Jin [6], Proposition 3.4).

Theorem 7.3 *If ${}^*\mathbb{R}$ is special (instead of fully saturated), the conclusions of Theorem 7.2 and Corollaries 7.1 and 7.2 still hold.*

Proof: Assume ${}^*\mathbb{R}$ is special, and let $\mu = \mathrm{card}({}^*\mathbb{R})$. Let $\{\mathcal{A}_\kappa : \kappa < \mu\}$ be a specializing chain for ${}^*\mathbb{R}$, and let any positive infinitesimal ρ be given. Since ${}^*\mathbb{R}$ is the union of the $\mathcal{A}_\kappa$'s, we can choose $\nu < \mu$ such that $\rho \in \mathcal{A}_\nu$. Now replace every $\mathcal{A}_\kappa$ with $\kappa < \nu$ by this $\mathcal{A}_\nu$. In this way we get a new specializing chain for ${}^*\mathbb{R}$ such that $\rho \in \mathcal{A}_\kappa$ for every $\kappa < \mu$. So, for every $\kappa < \mu$, we can construct quotient structures from $\mathcal{A}_\kappa$ that are analogous to ${}^\rho\mathbb{R}$ and $\widehat{{}^\rho\mathbb{R}}$, with

the constructions combined into one step as in the proof of Theorem 7.2. That is, let

$$D_\kappa = \{x \in \mathcal{A}_\kappa : |x| < \rho^{-1/n}, \text{for every } n \in \mathbb{N}\},$$

and let

$$K_\kappa = \{x \in \mathcal{A}_\kappa : |x| < \rho^{1/n}, \text{for some } n \in \mathbb{N}\}.$$

As before, K_κ is the maximal convex ideal of the convex ring D_κ. Let $\mathcal{B}_\kappa$ be the quotient field D_κ/K_κ, and let Q_κ be the quotient map. There is a canonical embedding of each $\mathcal{B}_\kappa$ in $\widehat{^\rho\mathbb{R}}$. Identifying each $\mathcal{B}_\kappa$ with its image in $\widehat{^\rho\mathbb{R}}$, it is simple to show that $\widehat{^\rho\mathbb{R}}$ is the union of the increasing sequence of subfields $\{\mathcal{B}_\kappa : \kappa < \mu\}$. Our next goal is to show that these subfields of $\widehat{^\rho\mathbb{R}}$ form a specializing chain for $\widehat{^\rho\mathbb{R}}$. By Theorem 7.1 (c), each $\mathcal{A}_\kappa \prec {}^*\mathbb{R}$. Therefore, since ${}^*\mathbb{R}$ is real-closed, so is each $\mathcal{A}_\kappa$. It follows, as in the proof of Theorem 4.1, (b) and (c), that each $\mathcal{B}_\kappa$ is real-closed. Thus, if $\kappa_1 < \kappa_2 < \mu$, the fact that $\mathcal{B}_{\kappa_1} \subseteq \mathcal{B}_{\kappa_2}$ implies that $\mathcal{B}_{\kappa_1} \prec \mathcal{B}_{\kappa_2}$. Also, since each $\mathcal{A}_\kappa$ is κ^+-saturated, the proof of Theorem 7.2 shows that each $\mathcal{B}_\kappa$ is κ^+-saturated as well. (I.e., the argument there that shows that any level of saturation satisfied by ${}^*\mathbb{R}$ is passed on to $\widehat{^\rho\mathbb{R}}$; full saturation is not required for this.) So we have shown that $\{\mathcal{B}_\kappa : \kappa < \mu\}$ is a specializing chain for $\widehat{^\rho\mathbb{R}}$, and therefore $\widehat{^\rho\mathbb{R}}$ is special. We know that $\text{card}(\widehat{^\rho\mathbb{R}}) \le \mu = \text{card}({}^*\mathbb{R})$. But for each cardinal $\kappa < \mu$, $\widehat{^\rho\mathbb{R}}$ contains the subfield $\mathcal{B}_\kappa$, which is κ^+-saturated and therefore has cardinality greater than κ. Therefore, $\text{card}(\widehat{^\rho\mathbb{R}}) = \text{card}({}^*\mathbb{R})$. As in the proof of Theorem 7.2, ${}^*\mathbb{R}$ and $\widehat{^\rho\mathbb{R}}$ are elementarily equivalent. Therefore, by Theorem 7.1 (d), ${}^*\mathbb{R}$ and $\widehat{^\rho\mathbb{R}}$ are isomorphic. The conclusions of Corollaries 7.1 and 7.2 follow immediately. ▲

Here is a brief summary of the main results of this section:

Corollary 7.3 *(a) There exists a ${}^*\mathbb{R}$ that satisfies the conclusions of Theorem 7.2 and Corollaries 7.1 and 7.2.*

(b) Assume GCH. Then, for every regular uncountable cardinal κ, there is a ${}^*\mathbb{R}$ of cardinality κ that satisfies the conclusions of Theorem 7.2 and Corollary 7.1, and, if $\kappa >$ card$(\mathbb{R})$, Corollary 7.2.

Proof: (a) Immediate by Theorem 7.3 and the remark preceding it.

(b) For any infinite cardinal κ, it is possible (using a so-called κ^+-good ultrafilter) to construct a ${}^*\mathbb{R}$ that is κ^+-saturated and has cardinality 2^κ.

(See T. Lindstrom [12], Theorem III.1.3.) So GCH implies that there exists a fully saturated ${}^*\mathbb{R}$ of each uncountable regular cardinality. ▲

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